The Application and Solutions of the Heat Equation

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Dedicated to my family: Patrick Yaluma, Irene Yaluma, and my four siblings Patricia Yaluma, George Yaluma, Andrew Yaluma and Chilemya Yaluma

Abstract

In this paper, I investigate some of the applications of the heat equation. I use the heat equation to determine the best arrangement or configuration of a system with two or more conducting materials of different thermal conductivities. In other words, I will compare the different arrangements and analyze their flow of heat. The idea is to be able to determine if the arrangements created produce a system that is a good or bad carrier of heat. The good carriers are good conductors or bad insulators and the bad carriers are the good insulators or bad conductors. In order to do so, I will solve the heat equation numerically using the Finite Difference Method (FDM). To make this project more realistic, I will work with the 3-D heat equation. Later, we discuss and analyze the results.
Contents

1 Introduction

2 Theory
  2.1 Transient Conduction
  2.2 Steady-State Conduction

3 The Boundary and Initial Conditions

4 Experimental Methods
  4.1 Heat Equation in 1-D
  4.2 Heat Equation in 2-D
  4.3 Heat Equation in 3-D

5 Results

6 Conclusions

7 Acknowledgments

8 Bibliography
1. Introduction

The heat equation is an important partial differential equation (PDE) which describes the distribution of heat (or variation in temperature) in a given region over time. For better understanding of this project, it is very important that we understand the difference between heat and temperature. Heat is a process of energy transfer as a result of temperature difference between the two points. Thus, the term 'heat' is used to describe the energy transferred through the heating process. Temperature, on the other hand, is a physical property of matter that describes the hotness or coldness of an object or environment. Therefore, no heat would be exchanged between bodies of the same temperature.

Suppose you have a function, $U(x, y, z, t)$, which describes the temperature of a conducting material at a given location, $(x, y, z)$, you can use this function to determine the temperature at any position on the material at a future time, $t+1$. The function $U$ changes over time as heat spreads throughout the material and the heat equation is used to determine this change in the function $U$. The gradient of $U$ describes which direction and at what rate is the temperature changing around a particular region of the material. Therefore, the gradient of temperature is the flow of heat through the material. This gradient will help us determine the flow of heat through various materials. This is analogous to the flow of water in a pipe.

2. Theory

The aim of this project is to be able to determine the flow of heat of various materials i.e. different thermal conductivities. Does the arrangement of conductors or insulators affect the rate at which the heat flows? Imagine a room with a wall that is made of different materials such as wood, metal or bricks arranged in different ways. The room is at room temperature, say 25°C and does not generate any heat (no air conditioner) and it is surrounded by the outside environment which has a temperature of 0°C. The room is so tiny relative to the outside environment therefore any heat flow from the room to the outside would not change the temperature outside. However, the temperature inside the room is prone to changes due to the surrounding temperature. How can we ensure that we maintain the room temperature for
the longest possible time without the use of an air conditioner? If the walls of the room are bad insulators of heat, it is almost impossible to maintain the room temperature. This is when it is important that we maximize the materials and knowledge that we have to build a wall that would keep the room temperature constant. It is possible that one can just buy building materials with low thermal conductivity. However, the constraints are that we have a variety of bad and good thermal conductors and we are trying to build the best configuration with the materials that we have.

To answer these questions, I have created materials with different thermal conductivities arranged in different ways. I am more interested in two cases: case 1) what happens to the heat flow when I reverse the order of thermal conductivities and case 2) what happens when I put the materials with high thermal conductivities on the edges or vice versa. To test these arrangements, I will set the temperature on one end of the material to be at 0°C and the other end at 100°C. But before we get into that, let us have a look at the two kinds of conduction that are important to the understanding of this project.

2.1 Transient Conduction

Transient Conduction as the name suggests is a form of conduction that is fleeting or lasts for only a short period of time. The other name for this type of conduction is non-steady state. In general, whenever temperatures are changing in time at any place within an object, thermal energy flow occurs due to transient conduction. Transient conduction normally occurs when there is a change in temperature at the boundaries of an object or may occur with temperature changes inside an object as a result of a new source or sink of heat that causes the change in temperatures. When equilibrium is reached, the heat flow into the system will equal the heat flow out of the system and temperatures at each point inside the system no longer vary. It is at this point that transient conduction ends and steady-state conduction may continue to occur if there is a continual flow of heat. If there are changes in external temperatures or the internal heat generation changes are too rapid for equilibrium of temperatures in space to take place, then the system never reaches a state of unchanging temperature distribution in time, and the system remains in a transient state.
2.2 Steady-State Conduction

Steady-state is a form of conduction that occurs when the temperature difference driving the conduction are kept constant. This means that after a certain time, the spatial distribution of temperatures in the conducting object does not change any further. This implies that all partial derivatives of temperature, $U(x, y, z, t)$, with respect to space are either zero or constants. However all the derivatives of temperature at any given point with respect to time are uniformly zero. In Steady-state conduction, the amount of heat entering any region of an object is equal to the amount of heat coming out. If this was not true the temperatures would be rising or falling. In steady-state conduction, all the laws of direct current electrical conduction can be applied to 'heat currents'.

3 The Boundary and Initial Conditions

To be able to solve the second-order partial differential heat equation in the spatial coordinates, we need to know the boundary conditions and the initial conditions. The boundary conditions specify how our system interacts with the outside surroundings. There are three general types of boundary conditions: Dirichlet, Neumann and Mixed boundary conditions.

3.1 Dirichlet Boundary Conditions

The Dirichlet boundary conditions say that the temperature is set at the boundary. In one dimensional system they take the form of

$$U(x_o, t) = U_{bc1}(t)$$

This says that at the left-hand-side ($x = 0$) boundary of our one-dimensional system, the temperature is a specified function of time. However, we can choose this temperature to be a constant, in which case we have

$$U(x_o, t) = U_{bc1}$$

If we set our boundary temperature to a constant value, we have a physical situation where our system is touching an infinite heat reservoir that maintains a constant temperature. In a one-dimensional system, we must have two boundary conditions, one at the left-hand-side boundary and the other
at the right-hand-side boundary. Since we have already set the temperature at the left-hand-side \( (x = 0) \), we must also set the temperature at the right-hand-side. So if our one-dimensional system is of length \( x_{\text{max}} \) in the \( x - \text{direction} \), then our second Dirichlet boundary condition is

\[
U(x_{\text{max}}, t) = U_{bc2}(t)
\]

or

\[
U(x_{\text{max}}, t) = U_{bc2}
\]

### 3.2 Neumann Boundary Conditions

The Neumann boundary conditions say that the heat flux is set at the boundaries. In one dimensional system they take the form of

\[
\frac{dU(x_o, t)}{dx} = \frac{dU(t)}{dx} \bigg|_{bc1}
\]

This says that at the left-hand-side boundary of our one-dimensional system, the heat flux is a specified function of time. If the heat flux is constant we have

\[
\frac{dU(x_o, t)}{dx} = \frac{dU}{dx} \bigg|_{bc1}
\]

If we keep the heat flux constant it means that we have a physical situation where our system is touching an infinite supply or heat (heat source) that maintains a constant flow of heat into the system regardless of the temperature. Note that if this were the case, the system would be in steady-state conduction mode. This would occur when one end of the wire or rod is well insulated and therefore no heat leaves it. The heat flux is zero \( \left( \frac{dU(x_o, t)}{dx} = 0 \right) \). Therefore our second Neumann boundary condition would be

\[
\frac{dU(x_{\text{max}}, t)}{dx} = \frac{dU(t)}{dx} \bigg|_{bc2}
\]

or

\[
\frac{dU(x_{\text{max}}, t)}{dx} = \frac{dU}{dx} \bigg|_{bc1}
\]
3.3 Mixed Boundary Conditions

The mixed boundary conditions are very relevant physical systems and as the name suggests, they are a mixture of Dirichlet and Neumann boundary conditions. In this project, I have experimented with all these boundary conditions but the results I will show are mostly from mixed boundary conditions. I have created a 3-D rectangular box of length \(x = 50\text{units}\), length \(y = 20\text{units}\) and length \(z = 10\text{units}\). The box has set temperatures at front and back sides (Dirichlet boundary conditions) and on all the other sides the heat flux is equal to zero (Neumann boundary conditions). This creates a well insulated conductor that can only lose heat through its front and back faces. Analogous to a well insulated (dc) current carrying wire. The mixed boundary conditions in one dimension have the form:

\[
\frac{dU(x_o, t)}{dx} + U(x_o, t) = U_{bc1}(t) + \frac{dU}{dx} |_{bc1} = f(t)
\]

3.4 Initial Conditions

The two types of initial conditions that I have explored in this project are Generalized Initial Conditions and Constant Temperature Initial Conditions. For generalized initial condition we need to know the temperature at every point in our system at time equals zero, \(t_o\). In general, this IC looks like the following in one dimension:

\[
U(x, t_o) = U_{ic}(x)
\]

This implies that the temperature every where on our system is a specified function of space. If we choose to set the temperature constant every where on our system, then our IC would look like the following

\[
U(x, t_o) = U_{ic}
\]

4 Experimental Methods

To solve the heat equation numerically, I will use the Finite Difference Method or FDM. The FDM can be used to solve or approximate the derivatives of some second-order boundary value problems. This method relies on discretizing a function on a grid. It cuts the structure or function in equal
spaces as shown below;

From the figure on the left, we can see that the function is cut into equal spaces of width, $h$. The figure on the right shows us exactly how all the points are connected to each other. So if we know the temperature at position $(j, n)$, we also know the temperature at position $(j - 1, n)$, $(j + 1, n)$ and $(j, n + 1)$.

### 4.1 Heat Equation in 1-D

The heat equation in one dimension is written as the following;

$$
\frac{\partial U}{\partial t} = \alpha \left( \frac{\partial^2 U}{\partial x^2} \right)
$$

(1)

where $U(x, t)$ is a function of temperature. In this case you can think of a one dimensional rectangular thin wire with length $x$. Ignore the width and height dimensionality. The one end of the length of the wire is set at $0^\circ$C whereas the other end is set at $0^\circ$C. These are its boundary conditions. We also need to specify the temperature at every position on the wire at time, $t_o$. (the initial conditions). To solve this one-dimensional heat problem we need to transform the above heat equation into an explicit method using the second-order central Finite Difference Method. Therefore, explicitly we
can write the one dimensional heat equation as

\[ \frac{U_{j}^{n+1} - U_{j}^{n}}{\tau} = \alpha \left( \frac{U_{j+1}^{n} - 2U_{j}^{n} + U_{j-1}^{n}}{h^2} \right). \]  

(2)

This equation can then be implemented and solved easily using Matlab or other languages. The value \( \frac{\alpha \tau}{h^2} \) should be much, much smaller than 1, otherwise you get unexpected errors. In this case, we assume that \( \alpha \) is a constant although later we are going to see that alpha could be defined as a function that depends on space. As we can see, the heat equation in 1-D explicit form is straightforward because the right hand side has only one term. For the 2-D and 3-D heat equations we are going to add the second and third terms respectively.

4.2 Heat Equation in 2-D

In two dimensions, we now have a structure or object that looks more like a flat sheet of paper. The two dimensional heat equation is therefore written with an additional term in \( y \) as follows;

\[ \frac{\partial U}{\partial t} = \alpha \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right). \]  

(3)

In this case, to solve the above equation, we are going to need four boundary conditions (for the four edges) and the initial condition that will specify the temperature of our system everywhere. Here I could either use two Dirichlet BCs and two Neumann BCs or make them all Neumann BCs. The IC could be some temperature function of space or just a constant.

4.3 Heat Equation in 3-D

The heat equation in three-dimensional space is more interesting and more practical. Now we are going to have a structure or object that looks like a rectangular box or rod with length \( x = 50 \text{units} \), length \( y = 20 \text{units} \) and length \( z = 10 \text{units} \). This object has six-faces or six-sides. Therefore, to be able to solve this problem I will need six boundary conditions. For this object I chose to have two Dirichlet BCs and four Neumann BCs. I set the front and back faces to 100°C and 0°C respectively. Then all the other sides were insulated, meaning heat flux was zero. This way I could limit the loss
of heat to the ends only (front and back). The Initial condition to specify temperature every where was a specified exponential function of space. This function is only dependent on position $x$ and therefore temperatures only vary in the $x$ direction. Our specified initial condition in 3-D is the following equation

$$U(x, y, z, t_0) = \frac{100}{e^{\frac{x-20}{3}} + 1}$$  \hspace{1cm} (4)

In 3-D, the heat equation with all the three terms is therefore

$$\frac{\partial U}{\partial t} = \alpha \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right)$$  \hspace{1cm} (5)

or

$$\frac{\partial U}{\partial t} = \nabla \cdot (\alpha \nabla U)$$  \hspace{1cm} (6)

Equation (6) tells us that we have to evaluate the gradient of $(\alpha \nabla U)$. This means that if $\alpha$ is a constant, we get

$$\frac{\partial U}{\partial t} = \alpha (\nabla^2 U)$$  \hspace{1cm} (7)

otherwise we get

$$\frac{\partial U}{\partial t} = \alpha (\nabla^2 U) + (\nabla \alpha) \nabla U$$  \hspace{1cm} (8)

In one of my systems, I defined $\alpha$ as a function of space, $\alpha(x, y, z)$, and I had to use equation (8) to obtain the correct heat equation.

5 Results

From the 3-D system that I described above, I am going to show some results and explain the plots. In addition, I will also show results from a 2-D system with two Dirichlet boundary conditions and two Neumann boundary conditions. To begin with, let us examine the temperature distribution of a 3-D system in which heat is flowing in only one direction, the $x$ direction. The plot of temperature, $U(x, y, z, t)$, against space, $(x, y, z)$, gives the following results. Note that from time $t_0$, to time $t_{\text{max}}$, the surface of the plots become more linear. When the temperature reaches its equilibrium, we expect to get a completely linear plot.
Figure 1: Temperature distribution at $t = t_0$

Figure 2: Temperature distribution at $t = 100$
Figure 3: Temperature distribution at $t = 200$

Figure 4: Temperature distribution at $t = 302$
From the above results it is clear that the surface plots are becoming linear as time passes. This is due to the fact that the system will reach some equilibrium temperature at which the heat flow is constant. The temperature difference driving the conduction are kept constant therefore the spatial distribution of temperatures in the conducting object does not change any further. All partial derivatives of temperature, $U(x, y, z, t)$, with respect to space are constants and the system has reached steady-state. Another distribution of temperature of a system that is configured differently as the one shown above is shown below. For this system, the thermal conductivities were constants arranged in such a way that we have a high $\alpha$ in the center and lower various $\alpha$s on the outside.

![Figure 5: Temperature distribution at $t = 100$](image)

![Figure 6: Temperature distribution at $t = 200$](image)
Figure 7: Temperature distribution at $t = 210$

The next set of results I would like to show are heat flow plots. Heat flow is the gradient of temperature. Therefore, these graphs are crucial because they tell us the rate at which heat flows through these systems. It is through analyzing these graphs that we can know whether a system is a good or bad carrier of heat. The first two heat flow plots are for a 2-D system with two Dirichlet BCs and two Neumann BCs.
Figure 8: Heat flow at position, $x = 30$ and time, $t$

Figure 9: Heat flow at position, $x = 30$ and time, $t$ but with reversed thermal conductivity
The next two heat flow plots are from a 3-D system (figure 1-3) in which the heat flow is restricted in only one direction, \( x - axis \). In this one, the thermal conductivities are reversed too.

Figure 10: Heat flow at position, \( x = 20 \) and \( time, t \)

Figure 11: Heat flow at position, \( x = 20 \) and \( time, t \) but with reversed thermal conductivity
It is clear that reversing the order of different thermal conductivities affect the flow of heat. From figure 8, we find the heat flow to be approximately 0.41 joules per second. The negative sign just gives the direction of heat flow. Figure 9 with reversed $\alpha$ gives about 0.81-0.83 joules per second. A much higher rate than figure 8. Although the heat flows in figure 10 and 11 do not seem to be very constant we can get the general idea that the heat flows are different when the thermal conductivities are reversed.

The last set of results that I would like to show are from the system much like the system in figures 5 - 7. The only difference here is that the thermal conductivities are arranged in two different cases; case 1) the weak thermal conductivities were placed on the edges with the strong one in the middle and case 2) the strong thermal conductivities were placed on the edges with the weak one in the middle. To analyze the flow of heat we are going to examine the flow of heat in the $x$ - direction. Note that in this system, I did not reverse the order of $\alpha$s but I simply rearranged them in two different cases. The results are obtained below. First off, I will show the temperature distribution of these two systems.
Figure 12: Temperature distribution weak $\alpha$ on edges

Figure 13: Temperature distribution strong $\alpha$ on edges
Now I am going to show the flow of heat for figures 12 and 13. Here are the plots:

Figure 14: Heat flow weak $\alpha$ on edges

Figure 15: Heat flow strong $\alpha$ on edges
Again, we can clearly see what happens here. The flow of heat for a system with weaker thermal conductivities on the edges is very slow, about 0.05 - 0.1 joules per second. This is significantly slower than the heat flow of a system with stronger thermal conductivities on the edges. For this system we get the rate of about 1.6 - 1.8 joules per second.

6 Conclusions

I have successfully used computational tools to solve the heat equation (PDE). I have also managed to translate a physical heat transfer problem into a partial differential equation with the use of boundary conditions and initial conditions. From the above results we can conclude that, it is important to know how you want to arrange your different materials before you can build your system otherwise you might end up with a system that is different from what you intended. The results in this paper show that reversing the different materials affect the flow of heat. In addition, if you want to have a system i.e. a room that does not lose much heat to the outside environment, you might want to have your best materials (weak $\alpha$) on the edges. Else if you want your system to be able to lose heat fast, you want to put your best materials (strong $\alpha$) on the edges.

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References