Large-N expansion for unitary superfluid Fermi gases

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We analyze strongly interacting Fermi gases in the unitary regime by considering the generalization to an arbitrary number N of spin-1/2 fermion flavors with Sp($2N$) symmetry. For $N\to\infty$ this problem is exactly solved by the Bardeen-Cooper-Schrieffer–Bose-Einstein condensate mean-field theory, with corrections small in the parameter $1/N$. The large-N expansion provides a systematic way to determine corrections to mean-field predictions, allowing the calculation of a variety of thermodynamic quantities at (and in proximity to) unitarity, including the energy, the pairing gap, and the upper-critical polarization (in the case of a polarized gas) for the normal to superfluid instability. For the physical case of $N=1$, among other quantities, we predict in the unitarity regime, the energy of the gas to be $\xi=0.28$ times that for the noninteracting gas and the pairing gap to be 0.52 times the Fermi energy.

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I. INTRODUCTION

A. Motivation and background

Recent advances in atomic gases near a Feshbach resonance (FR) have led to the experimental realization of resonantly paired superfluidity of fermionic atomic gases [1–6]. Although atomic interactions are generally weak, such superfluidity has been achieved using the strong attractive interactions provided by proximity to a magnetic-field tuned Feshbach resonance between the two hyperfine levels (isomorphic to spin-$\frac{1}{2}$ states) undergoing pairing. Indeed, the tunability of such Feshbach resonances, experimentally controllable via an external magnetic field, allows unprecedented access to a wide range of fermionic interaction strengths characterized by the vacuum s-wave scattering length a_s that diverges for a resonance tuned to zero energy. As the FR detuning $\delta\propto -1/a_s$ is varied, the character of superfluidity evolves from a weakly paired Bardeen-Cooper-Schrieffer (BCS) regime at large positive FR detuning (where $a_s<0$) to a strongly paired molecular Bose-Einstein condensate (BEC) regime at large negative detuning (where $a_s>0$). The experimental signatures of such pairing and superfluidity in ultracold gases of ^{40}K and ^{6}Li have included direct measurements of the condensate density [1,2] and the pairing gap [6,7] and the observation of vortex lattices in rotating clouds [8].

From a theoretical perspective, a quantitative description of such resonantly interacting Fermi superfluids is well developed away from the resonance, where a_s is short compared to atom spacing. Then, a controlled perturbative expansion in a natural small parameter $|a_s|n^{1/3}\ll 1$ (with n the fermion density) allows a quantitative theoretical analysis of both the deep BCS [9,10] and deep BEC [11–13] regimes. However, the aforementioned present-day experiments are typically in the crossover between the BCS and BEC regimes, where $|a_s|n^{1/3}\gg 1$. Thus, the absence of a small parameter [14–16] near the resonance precludes a systematic perturbative expansion for a quantitative description of this theoretically challenging regime.

A particular point of interest is the so-called unitarity point, precisely at zero FR detuning, at which $a_s^{-1}=0$ and fermion scattering is characterized by a maximum scattering phase shift of $\pi/2$. At this point, the system does not contain any scale besides the Fermi wavelength k_F^{-1} set by the atom density $n=k_F^3/(3\pi^2)$, and the only energy scale is the Fermi energy $\epsilon_F=\hbar^2k_F^2/2m$ and therefore the free energy, and any quantities related to it, is given by ϵ_F multiplied by a *universal* dimensionless function of k_BT/ϵ_F [17]. In particular, this implies that, at zero temperature, the internal energy per particle ϵ is simply proportional to that of a noninteracting Fermi gas

$$\epsilon=\xi\tfrac{3}{5}\epsilon_F. \tag{1.1}$$

Recently, there has been much theoretical interest in computing ξ and other universal parameters, motivated by the possibility of attaining a quantitative understanding of unitary quantum gases as realized in cold-atom experiments but also having applications to nuclear physics and astrophysical systems such as neutron stars [18–29]. The purpose of this paper is to provide a theoretical framework for a *systematic* determination of the value of such universal parameters at unitarity. Our method is based on the introduction of an artificial small parameter $1/N$, with N the number of distinct "spin"-$\frac{1}{2}$ fermion flavors with Sp($2N$) symmetry. We then use it to extrapolate to the experimentally relevant case of a single flavor ($N=1$) of two opposite-"spin"-$\frac{1}{2}$ (hyperfine levels) fermionic atoms.

The motivation for such a generalization is that, for $N\to\infty$, the problem may be solved exactly [30], with the solution taking the form of the standard BCS mean-field theory. We can then compute corrections in the small parameter $1/N$, obtaining a systematic expansion (in principle to arbitrary order) about this solvable limit. Since it is believed that there are no phase transitions with decreasing N (the large-N solution having the same broken symmetry as the exact ground state at $N=1$), we expect the large-N results to be smoothly connected to the physical case of $N=1$. The large-

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N expansion may be thought of as a way to systematically organize corrections around the well-known BCS mean-field solution. Since this $N\to\infty$ solution is known to give a relatively good estimate, we expect the $1/N$ expansion to converge rapidly.

To further motivate our study, it is instructive to briefly review other theoretical approaches. As we have noted, near the unitarity point a *quantitative* theoretical understanding of this strongly correlated problem is limited by the absence of any physical small parameter [14,16]. However, since, the thermodynamic quantities are expected to evolve smoothly from the BCS to the BEC limits, a *qualitative* description has come from various uncontrolled (by any small parameter) schemes, that interpolate between these two regimes. Starting from a variational wave function (equivalent to mean-field theory), Eagles [9] and Leggett [10] studied the crossover from the BCS to the BEC limit at zero temperature. Nozieres and Schmitt-Rink studied the same problem at finite temperature, taking into account the attraction of fermions beyond mean-field theory [31]. The repulsion between bound pairs was first worked out using a functional integral formalism [32,33] and a self-consistent theory [34]. However, except for numerical Monte Carlo calculations [19–22], a full description of the system around unitarity in a *quantitative* fashion has remained elusive due to the absence of a small parameter.

More recently, stimulated by progress made in the theory of critical phenomena (where, similarly, no physical small parameter exists near a critical point) [35], progress has been made by studying the system in d dimensions, treating the unitary regime in a systematic expansion in a small parameter associated with the dimensionality of space [23–25,36].

The large-N approach used here is close in spirit, but is complementary to such an expansion-in-dimension study, in that we also introduce an artificial, but distinct small parameter and perform a systematic perturbative expansion in it. Such an approach has also been extremely successful in a variety of field theory and statistical physics contexts [37–39], applied to a description of continuous phase transitions close to a critical point.

B. Summary of results

Our results are predictions for the lowest nontrivial order in $1/N$ for an array of physical quantities at and around the unitary point. At unitarity $a_s^{-1}=0$, a Fermi gas at density n per fermionic flavor is characterized by a single energy scale $\epsilon_F=(3\pi^2 n)^{2/3}/(2m)$. In this regime, we find that, in the symmetry-broken (paired-superfluid) phase at $T=0$, the chemical potential μ, the order parameter Δ, and the excitation gap Δ_{exc} to order $1/N$ are given by

$$\Delta/\epsilon_F = 0.6864 - 0.163/N + O(1/N^2), \tag{1.2}$$

$$\Delta_{\text{exc}}/\epsilon_F = 0.6864 - 0.196/N + O(1/N^2), \tag{1.3}$$

$$\mu/\epsilon_F = 0.5906 - 0.312/N + O(1/N^2), \tag{1.4}$$

with the first $N\to\infty$ term corresponding to the well-known mean-field theory result [9]. As we discuss in the Appendix, although $\Delta=\Delta_{\text{exc}}$ in the $N=\infty$ limit, at subleading order these quantities differ, as seen in Eq. (1.3).

By using scaling arguments, it can be shown that, at unitarity, the ratio μ/ϵ_F is equal to ξ, defined by Eq. (1.1). Substituting $N=1$ in Eqs. (1.2) and (1.4) gives our predictions

$$\Delta/\epsilon_F = 0.523, \tag{1.5}$$

$$\Delta_{\text{exc}}/\epsilon_F = 0.490, \tag{1.6}$$

$$\xi = 0.279. \tag{1.7}$$

We have also computed the critical temperature T_c and the chemical potential μ at T_c to be given by

$$k_B T_c/\epsilon_F = 0.4964 - 1.31/N + O(1/N^2), \tag{1.8}$$

$$\mu/\epsilon_F = 0.7469 - 0.58/N + O(1/N^2). \tag{1.9}$$

The $1/N$ expansion can also be applied to the problem of a unitary Fermi gas with population imbalance $\delta n=n_\uparrow-n_\downarrow$ between the densities $n_\uparrow, n_\downarrow$ of the two hyperfine fermion components, that has been realized in experiments [40–42] and has been a focus of intense theoretical activity; see, e.g., Ref. [43] and references therein. Mean-field theory predicts [43–46], that, for arbitrarily small polarization $P\equiv\delta n/n$, a Fermi gas in the unitary and positive-detuning regimes phase separates into a polarized normal (unpaired) phase and an unpolarized paired-superfluid phase, with the average density and polarization equal to the experimentally imposed values. Such phase separation has, in fact, already been observed experimentally. We find that, within the $1/N$ expansion (consistent with mean-field theory) at unitarity, the paired superfluid is unstable to phase separation for an arbitrarily small polarization, i.e., $P_{c1}=0$ [43].

For sufficiently large P, or, equivalently, sufficiently large applied "Zeeman field" $h=\frac{1}{2}(\mu_\uparrow-\mu_\downarrow)$ a uniform polarized normal phase is stable and the corresponding critical values P_{c2} and h_{c2} are also universal quantities at unitarity. We find their $1/N$ expansion to be given by

$$h_{c2}/\epsilon_F = 0.6929 + 0.087/N + O(1/N^2), \tag{1.10}$$

$$P_{c2} = 0.9326 - 0.631/N + O(1/N^2), \tag{1.11}$$

$$\mu/\epsilon_F = 0.8585 - 0.458/N + O(1/N^2), \tag{1.12}$$

where the chemical potential is evaluated at h_{c2}.

This paper is organized as follows. In Sec. II, we introduce the one-channel model and its generalization to the N flavor case. In Sec. III, we show that the solution for $N\to\infty$ is equivalent to the mean-field solution of the $N=1$ model. We then take into account, systematically, the fluctuation corrections to leading order in $1/N$ to the internal energy, excitation gap, the critical temperature and the upper-critical polarization, and extend our results to the vicinity of the unitary point. In Sec. IV, we discuss our results and compare them to recent measurements and predictions of other theoretical approaches. In Sec. V, we provide a brief summary.

II. MODEL

We study a three-dimensional resonant Fermi gas confined to a box of size $L\times L\times L=V$, described by a Hamiltonian [14] (setting $\hbar=1$)

$$\mathcal{H}=\sum_{\sigma=\uparrow,\downarrow}\int d^3\mathbf{r}\psi_\sigma^\dagger(\mathbf{r})\left(-\frac{\nabla_{\mathbf{r}}^2}{2m}-\mu_\sigma\right)\psi_\sigma(\mathbf{r})+\lambda\int d^3\mathbf{r}\psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow^\dagger(\mathbf{r})\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r}),\tag{2.1}$$

where $\lambda<0$ is the fermionic interaction and $\psi_\sigma^\dagger(\mathbf{r}),\psi_\sigma(\mathbf{r})$ are, respectively, the fermion creation and annihilation operators at position $\mathbf{r}$ and hyperfine state ("spin"-1/2) σ, which obey the usual anticommutation relation

$$\{\psi_\sigma(\mathbf{r}),\psi_{\sigma'}^\dagger(\mathbf{r}')\}=\delta(\mathbf{r}-\mathbf{r}')\delta_{\sigma,\sigma'}.$$

The chemical potential μ_σ fixes the average density n_σ of particles with spin σ. For simplicity we now set $\mu_\uparrow=\mu_\downarrow=\mu$, although later we shall allow the possibility of a nonzero chemical potential difference in order to study a polarized Fermi gas.

The partition function $Z=\mathrm{Tr}(e^{-\beta\mathcal{H}})$ gives the free-energy density

$$f=-\frac{1}{\beta V}\ln Z,\tag{2.2}$$

where $\beta=1/T$ is the inverse temperature, taking $k_B=1$ throughout.

The two-particle scattering length a_s is related to the strength of the atomic interaction λ via the relation

$$\frac{m}{4\pi a_s}=\frac{1}{\lambda}+\frac{1}{V}\sum_{\mathbf{k}}\frac{1}{2\epsilon_{\mathbf{k}}},\tag{2.3}$$

where $\epsilon_{\mathbf{k}}=k^2/(2m)$ is the free fermion dispersion (kinetic energy), taking $\hbar=1$ throughout. In the second term on the right-hand side of Eq. (2.3) an ultraviolet cutoff $\Lambda\sim 1/r_o$ is implied, set by the effective interaction range $r_o\ll 1/k_F$ [14,16].

We now proceed with the generalization of Eq. (2.1) to a model of N fermion flavors,

$$\mathcal{H}=\sum_{i=1}^{N}\sum_{\sigma=\uparrow,\downarrow}\int d^3\mathbf{r}\psi_{i\sigma}^\dagger(\mathbf{r})\left(-\frac{\nabla_{\mathbf{r}}^2}{2m}-\mu\right)\psi_{i\sigma}(\mathbf{r})+\frac{\lambda}{N}\sum_{i,j=1}^{N}\int d^3\mathbf{r}\psi_{i\uparrow}^\dagger(\mathbf{r})\psi_{i\downarrow}^\dagger(\mathbf{r})\psi_{j\downarrow}(\mathbf{r})\psi_{j\uparrow}(\mathbf{r}),\tag{2.4}$$

that is clearly equal to Eq. (2.1) for the case of $N=1$ and possessing invariance under the symplectic group Sp($2N$) (see Ref. [47]). As discussed in Sec. I A, the benefit of this expansion to N fermion flavors is that an exact solution may be obtained in the large-N limit. To see this, we write the partition function in terms of an imaginary-time coherent-state functional integral

$$Z=\int D\bar{\psi}_{i\sigma}(x)D\psi_{i\sigma}(x)\exp(-S)\tag{2.5}$$

with $\bar{\psi}_{i\sigma}(x),\psi_{i\sigma}(x)$ Grassmann fields labeling the corresponding coherent state at a space-time point $x=(\mathbf{r},\tau)$, and with the imaginary-time action S given by

$$S=\sum_{i=1}^{N}\sum_{\sigma=\uparrow,\downarrow}\int_0^\beta d\tau\int d^3\mathbf{r}\bar{\psi}_{i\sigma}(x)\left(\partial_\tau-\frac{\nabla_{\mathbf{r}}^2}{2m}-\mu\right)\psi_{i\sigma}(x)+\frac{\lambda}{N}\sum_{i,j=1}^{N}\int_0^\beta d\tau\int d^3\mathbf{r}\bar{\psi}_{i\uparrow}(x)\bar{\psi}_{i\downarrow}(x)\psi_{j\downarrow}(x)\psi_{j\uparrow}(x).\tag{2.6}$$

We decouple the atomic interaction via a standard Hubbard-Stratonovich transformation, that relies on the Gaussian integral

$$e^{-(\lambda/N)\phi^*(x)\phi(x)}=\frac{-N}{\pi\lambda}\int db(x)db^*(x)\exp\left(\frac{N}{\lambda}b^*(x)b(x)+b(x)\phi^*(x)+b^*(x)\phi(x)\right),\tag{2.7}$$

which we shall utilize with $\phi(x)$ given by a superposition of N flavors of bilinear fermionic fields

$$\phi(x)=\sum_{i=1}^{N}\psi_{i\downarrow}(x)\psi_{i\uparrow}(x).\tag{2.8}$$

With this transformation, the fermion fields appear quadratically and can therefore be formally integrated out. The partition function is then given by a functional integral over the boson field $b(x)$,

$$Z=Z_b^{-1}\int Db(x)Db^*(x)e^{-S(b)},\tag{2.9}$$

with the effective bosonic action given by

$$S(b)=-N\int_0^\beta d\tau\int d^3\mathbf{r}\frac{b^*(x)b(x)}{\lambda}-N\,\mathrm{Tr}\ln\{-G^{-1}[b(x),b^*(x)]\},\tag{2.10}$$

where the trace is over space, imaginary time, hyperfine, and flavor states, the Green's function (written in Nambu representation for the fermions) is

$$G^{-1}(x)=\begin{pmatrix}-\partial_\tau+\dfrac{\nabla^2}{2m}+\mu & b(x)\\ b^*(x) & -\partial_\tau-\dfrac{\nabla^2}{2m}-\mu\end{pmatrix},\tag{2.11}$$

and the Hubbard-Stratonovich normalization factor $Z_b=\int Db(x)Db^*(x)\exp(N\int_0^\beta d\tau\int d^3\mathbf{r}b^*(x)b(x)/\lambda)$. Note that in the limit where $N=1$ the effective action reduces to the one-channel model derived by Sá de Melo *et al.* [33].

We use this formulation to compute a number of thermodynamic quantities that characterize the resonant unitary Fermi gas. To this end, we observe that in the limit $N\to\infty$, these expectations values are dominated by the saddle point

of the functional integral. This is the origin of the aforementioned claim of an exact solution in the limit of large-N. As is well known, such a saddle-point evaluation amounts to the standard BEC-BCS mean-field approximation to the one-channel model [10] that is only *approximately* valid for the case $N=1$. At large N, fluctuations around this saddle point are small in the parameter $1/N$, generating $O(1/N)$ corrections, that organize into a systematic expansion in powers of $1/N$ for the partition function and physical quantities derived from it.

By representing the bosonic Green's function by a line, the partition function is given by a series of closed loops. The higher order terms in $1/N$ can be classified according to the number of loops for a particular diagram. Thus the $1/N$ expansion is equivalent to a so-called loop expansion.

The implementation of the loop expansion is performed around the (possibly complex) saddle point Δ, where Δ is the mean-field value of $b(x)$. At low temperature, that is our focus here, the saddle point yields a nonzero value for Δ. The complex field fluctuations $\hat{b}(x)$ around the saddle point are defined as

$$\frac{1}{\sqrt{N}}\hat{b}(x) \equiv b(x) - \Delta. \tag{2.12}$$

The Green's function matrix can be formally separated into the saddle point and fluctuation contributions,

$$G^{-1}(x) = G_{(0)}^{-1}(x) + \frac{1}{\sqrt{N}}G_{(1)}^{-1}(x), \tag{2.13}$$

where

$$G_{(0)}^{-1}(x) = \begin{pmatrix} -\partial_\tau + \dfrac{\nabla^2}{2m} + \mu & \Delta \\ \Delta^* & -\partial_\tau - \dfrac{\nabla^2}{2m} - \mu \end{pmatrix} \tag{2.14}$$

and

$$G_{(1)}^{-1}(x) = \begin{pmatrix} 0 & \hat{b}(x) \\ \hat{b}^*(x) & 0 \end{pmatrix}, \tag{2.15}$$

The bosonic action $S(b)$ can be expanded in powers of $1/N$,

$$\frac{S(b)}{N} = -\beta V\frac{|\Delta|^2}{\lambda} - \mathrm{Tr}[\ln(-G_{(0)}^{-1})] - \int_0^\beta d\tau \int d^3\mathbf{r}\left(\frac{1}{\sqrt{N}}\frac{\Delta\hat{b}^*(x) + \Delta^*\hat{b}(x)}{\lambda} + \frac{1}{N}\frac{|\hat{b}(x)|^2}{\lambda}\right) + \sum_{m=1}^{\infty}\frac{(-1)^m}{m}\frac{1}{N^{m/2}}\mathrm{Tr}(\{G_{(0)}G_{(1)}^{-1}[\hat{b}(x)]\}^m). \tag{2.16}$$

Up to this point, all the transformations of the partition function are exact. An approximation is made when only a finite number of terms in the action are considered, and the small parameter $1/N$ provides a systematic way to organize this expansion. To analyze the problem to lowest nontrivial order in $1/N$, it is sufficient to include terms up to $m=2$ in the action, yielding an approximate bosonic action

$$\frac{S(b)}{N} = S^{(0)} + \frac{1}{N}S^{(1/N)} + \cdots, \tag{2.17}$$

where

$$S^{(0)} = -\beta V\frac{|\Delta|^2}{\lambda} - \mathrm{Tr}[\ln(-G_{(0)}^{-1})] \tag{2.18}$$

and

$$S^{(1/N)} = -\int_0^\beta d\tau \int d^3\mathbf{r}\frac{|\hat{b}(x)|^2}{\lambda} + \frac{1}{2}\mathrm{Tr}\{G_{(1)}^{-1}[\hat{b}(x)]G_{(0)}G_{(1)}^{-1}[\hat{b}(x)]G_{(0)}\}, \tag{2.19}$$

that we shall analyze in the remainder of the paper. We note that, as usual, terms linear in $\hat{b}$ (the $m=1$ terms) automatically vanish by a virtue of the definition of the saddle point as the extremum of the action $S^{(0)}$.

III. LARGE-N EXPANSION

In the present section, we use the results of Sec. II to construct an expansion of the free-energy density f to leading order in small $1/N$ of the form

$$f = Nf^{(0)} + f^{(1/N)} + \cdots, \tag{3.1}$$

where the leading-order term $f^{(0)}$ comes from a saddle-point approximation to the bosonic functional-integral expression, Eq. (2.9), for the partition function.

A. Saddle-point approximation: Mean-field level

In this section, we compute the lowest order approximation to the free-energy density, namely the first term of Eq. (3.1), that is exact in the $N\to\infty$ limit. This corresponds to an evaluation of the free-energy density within the saddle-point approximation $f^{(0)} = -\frac{1}{\beta V}\ln Z^{(0)} = \frac{1}{\beta V}S^{(0)}$, which gives

$$f^{(0)} = -\frac{|\Delta|^2}{\lambda} - \frac{1}{V}\sum_{\mathbf{k}}(E_{\mathbf{k}} - \xi_{\mathbf{k}}) - \frac{2}{\beta V}\sum_{\mathbf{k}}\ln(1 + e^{-\beta E_{\mathbf{k}}}), \tag{3.2}$$

where we included an overall constant in the second term so that for $\Delta\to 0$, $f^{(0)}$ reduces to the free-energy density of a free Fermi gas. Here, $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta|^2}$ is the spectrum of the quasiparticles and $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$. Notice that this $N\to\infty$ limit yields a result for the free energy (and therefore all subsequent thermodynamics quantities) that is identical to the usual mean-field treatment based on a BCS ansatz for the ground-state wave function.

The gap equation, determined by minimizing the free energy with respect to the gap, $\frac{\partial f}{\partial \Delta} = 0$, yields

$$-\frac{m\Delta}{4\pi a_s} = \frac{\Delta}{V}\sum_{\mathbf{k}}\left(\frac{\tanh(\beta E_{\mathbf{k}}/2)}{2E_{\mathbf{k}}} - \frac{1}{2\epsilon_{\mathbf{k}}}\right). \tag{3.3}$$

The total atom density is $n\times N$, where n is the atom density of each of the fermion flavors. It is determined through a

Lagrange multiplier via $nN=-\frac{\partial f}{\partial \mu}$, which yields for $N\to\infty$,

$$n=\frac{1}{V}\sum_{\mathbf{k}}\left(1-\frac{\xi_k}{E_{\mathbf{k}}}\tanh(\beta E_{\mathbf{k}}/2)\right). \tag{3.4}$$

This number equation, along with Eq. (3.3), describes a resonant Fermi gas within the mean-field approximation.

The solution to these equations at zero and finite temperature has been discussed at length in, e.g., Refs. [10,14,31,33,48]. For completeness, below we review this mean-field solution near unitarity at zero temperature, near the transition temperature T_c, and its generalization to an imposed polarization [43].

1. Zero-temperature limit

At zero temperature, the mean-field ground state at unitarity, $(k_F a_s)^{-1}=0$, is a superfluid characterized by the BCS wave function, with the gap and chemical potential given by

$$\Delta_o^{(0)}/\epsilon_F=0.6864, \tag{3.5}$$

$$\mu_o^{(0)}/\epsilon_F=0.5906, \tag{3.6}$$

simply obtained by solving Eqs. (3.3) and (3.4) at $T=0$. Here, the superscript and subscript refer to the $1/N=0$ and $T=0$ limits, respectively.

2. Near T_c

At finite temperatures, a second-order superfluid-to-normal transition takes place at a critical temperature T_c, that in the $N\to\infty$ limit is given by

$$k_B T_c^{(0)}/\epsilon_F=0.4965, \tag{3.7}$$

$$\mu_c^{(0)}/\epsilon_F=0.7469, \tag{3.8}$$

obtained by solving these equations at $\Delta=0^+$; $\mu_c^{(0)}$ is the chemical potential at the transition as determined by the fixed-density condition Eq. (3.4).

3. Polarized Fermi gas

We can also study a polarized Fermi gas (one with population imbalance between the spin components) in the saddle-point approximation, achieved through coupling the fermions to an applied "Zeeman" field. Physically, this corresponds to imposing different chemical potentials, $\mu_{\uparrow/\downarrow}=\mu\pm h$ for the up and down spins, and yields the free-energy density

$$f^{(0)}=-\frac{|\Delta|^2}{\lambda}-\frac{1}{V}\sum_{\mathbf{k}}(E_{\mathbf{k}}-\xi_{\mathbf{k}})-\frac{1}{\beta V}\sum_{\sigma=\pm}\sum_{\mathbf{k}}\ln(1+e^{-\beta(E_{\mathbf{k}}+\sigma h)}). \tag{3.9}$$

The detailed mean-field phase diagram found by minimizing $f^{(0)}$ has been discussed in detail elsewhere [43,46]. The relevant parameters characterizing the phase diagrams are the detuning and h, or equivalently (and more directly related to experiments at fixed species imbalance) as a function of an imposed polarization

$$P=\frac{m}{n}=\frac{\dfrac{\partial f}{\partial h}}{\dfrac{\partial f}{\partial \mu}}, \tag{3.10}$$

with the density imbalance m and density n. We are particularly interested in the region near unitarity, where for sufficiently large h at fixed μ there is a first-order phase transition between the superfluid and normal states. At fixed imposed density n, the first-order transition opens up into a regime $h_{c1}<h<h_{c2}$ of phase separation, where on the BCS side and around the unitary point a gapped unpolarized superfluid and a polarized Fermi gas coexist. For $h<h_{c1}$, the system is in the homogeneous fully paired superfluid state. At unitarity, the lower-critical polarization for entering the regime of phase separation is $P_{c1}=0$ [43]. For $h>h_{c2}$, the normal state is stable. The instability to phase separation from the normal state takes place at $h=h_{c2}$ when the free energies of the normal (n) and superfluid (s) states cross, subject to the constraint on μ that the normal-state number equation is satisfied. Thus, we equate $f_s=f_n$, using the $N\to\infty$ approximation to the superfluid-state free-energy density $f_s\approx f^{(0)}(\Delta)$ (with Δ determined by the minimization $\partial f_s/\partial\Delta=0$), and the normal state free-energy density given by $f_n=f_s(0)$. Solving these equations at zero temperature, we find μ, h, and Δ at the transition (in agreement with earlier work [43,46])

$$\mu_{c2}^{(0)}/\epsilon_F=0.8586,$$

$$h_{c2}^{(0)}/\epsilon_F=0.6930,$$

$$\Delta_{c2}^{(0)}/\epsilon_F=0.9979. \tag{3.11}$$

The upper-critical polarization is given by combining the normal-state polarization Eq. (3.10) with f given by f_n (taking $T=0$),

$$P=\frac{(1+h/\mu)^{3/2}-(1-h/\mu)^{3/2}}{(1+h/\mu)^{3/2}+(1-h/\mu)^{3/2}}, \tag{3.12}$$

with the location of the transition defined by Eq. (3.11), yielding the mean-field result

$$P_{c2}=0.932\,61, \tag{3.13}$$

at unitarity.

The quantities that we have computed in this section within the $N\to\infty$ limit (that is equivalent to the BCS mean-field theory), are *universal*, i.e., independent of the microscopic interactions and can be obtained as derivatives of a universal free energy that is given by the Fermi energy (set by the atomic density n) times a system-independent scaling function of dimensionless variables such as T/ϵ_F and h/ϵ_F. For example, taking $h=0$ for simplicity, based on dimensional grounds, the free-energy density at unitarity can only depend on T and the density n [which we represent through ϵ_F, see Eq. (1.1)]

$$f(T,n) = \frac{3}{5}\frac{(3\pi^2)^{2/3}}{2m} n^{5/3} g\left(\frac{k_B T}{(3\pi^2)n^{2/3}/2m}\right), \quad (3.14)$$

where $g(x)$ is a dimensionless function. At zero temperature, the internal energy per particle [quoted in Eq. (1.1)] $\epsilon = f(T=0)/n$ is $\frac{3}{5}g(0)\epsilon_F$, while the chemical potential $\mu = \frac{\partial f}{\partial n}$ is equal to $\mu = g(0)\epsilon_F$. This leads to $\epsilon = \frac{3}{5}\mu$, that together with Eq. (3.6) gives our lowest order $N \to \infty$ (mean-field) estimate of the parameter ξ,

$$\xi = g(0) \approx 0.5906. \quad (3.15)$$

Other quantities that are connected to ξ at $T=0$ are the bulk modulus $B \equiv n^2 \partial^2 f / \partial n^2$ given by

$$B = \tfrac{2}{3}\xi n \epsilon_F, \quad (3.16)$$

and the first (isothermal) sound $v = \sqrt{B/nm}$.

In a similar fashion, one can derive the equation of state of the system and determine relations between the entropy, pressure, and internal energy [49]. Since these relations follow from scaling arguments, they hold for any number of fermion flavors. Consequently, these relations are expected to be preserved order by order in the $1/N$ expansion.

Having obtained leading-order results in $1/N$ (already reported in the literature as they correspond to the standard mean-field theory, exact for the $N=\infty$ model), in the next section we proceed to calculate the leading-order corrections in small $1/N$. We shall see that the large-N expansion of the one-channel model organizes subleading corrections to mean-field theory, essentially amounting to the random-phase approximation.

B. Leading order in $1/N$: Random phase approximation

We now consider the leading-order corrections in $1/N$ to the results of the preceding section, which requires an evaluation of the subleading contribution to the free energy. This is given by a Gaussian integration over $\hat{b}$ fluctuations around the saddle point, governed by the effective action Eq. (2.19), explicitly given by

$$S^{(1/N)} = \frac{1}{2}\sum_q \begin{pmatrix} \hat{b}^*(q) & \hat{b}(-q) \end{pmatrix} \begin{pmatrix} A(q) & B(q) \\ B^*(q) & A(-q) \end{pmatrix} \begin{pmatrix} \hat{b}(q) \\ \hat{b}^*(-q) \end{pmatrix}, \quad (3.17)$$

where $q = (\mathbf{q}, \Omega_\ell)$ and the bosonic Matsubara frequency $\Omega_\ell = 2\pi\ell/\beta$ with ℓ an integer. The matrix elements of the polarization matrix are

$$A(q) = -\frac{1}{\lambda} + \frac{1}{\beta V}\sum_k (G_{(0)}(k+q))_{11}(G_{(0)}(k))_{22}, \quad (3.18)$$

$$B(q) = \frac{1}{\beta V}\sum_k (G_{(0)}(q+k))_{21}(G_{(0)}(k))_{21}, \quad (3.19)$$

which satisfy the relations $A(q) = A^*(-q)$, $B(q) = B(-q)$.

The quantity $G_{(0)}$ appearing in Eqs. (3.18) and (3.19) is the standard saddle-point approximation to the single-particle Green's function in the BCS state

$$G_{(0)}(k) = \frac{-1}{\omega_\ell^2 + E_{\mathbf{k}}^2}\begin{pmatrix} i\omega_\ell + \xi_{\mathbf{k}} & -\Delta \\ -\Delta^* & i\omega_\ell - \xi_{\mathbf{k}} \end{pmatrix} = \begin{pmatrix} \frac{u_{\mathbf{k}}^2}{i\omega_\ell - E_{\mathbf{k}}} + \frac{v_{\mathbf{k}}^2}{i\omega_\ell + E_{\mathbf{k}}} & -\frac{u_{\mathbf{k}}v_{\mathbf{k}}}{i\omega_\ell - E_{\mathbf{k}}} + \frac{u_{\mathbf{k}}v_{\mathbf{k}}}{i\omega_\ell + E_{\mathbf{k}}} \\ -\frac{u_{\mathbf{k}}v_{\mathbf{k}}}{i\omega_\ell - E_{\mathbf{k}}} + \frac{u_{\mathbf{k}}v_{\mathbf{k}}}{i\omega_\ell + E_{\mathbf{k}}} & \frac{v_{\mathbf{k}}^2}{i\omega_\ell - E_{\mathbf{k}}} + \frac{u_{\mathbf{k}}^2}{i\omega_\ell + E_{\mathbf{k}}} \end{pmatrix}, \quad (3.20)$$

where

$$u_{\mathbf{k}} = \sqrt{\frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)}, \quad (3.21)$$

$$v_{\mathbf{k}} = \sqrt{\frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)}, \quad (3.22)$$

are the usual BCS coherence factors and $k = (\mathbf{k}, i\omega_\ell)$ and $\omega_\ell = (2\ell+1)\pi/\beta$ are the fermionic Matsubara frequencies and for simplicity we have taken Δ to be real. Evaluation of these normal and anomalous particle-particle bubbles yield, respectively,

$$A(q) = -\frac{1}{\lambda} + \frac{1}{2V}\sum_{\mathbf{k}} \Bigg[[\tanh(\beta E_{\mathbf{k}}/2) + \tanh(\beta E_{\mathbf{k+q}}/2)] \times \left(-\frac{u_{\mathbf{k+q}}^2 u_{\mathbf{k}}^2}{E_{\mathbf{k+q}} + E_{\mathbf{k}} - i\Omega_\ell} - \frac{v_{\mathbf{k+q}}^2 v_{\mathbf{k}}^2}{E_{\mathbf{k+q}} + E_{\mathbf{k}} + i\Omega_\ell} \right) + [\tanh(\beta E_{\mathbf{k}}/2) - \tanh(\beta E_{\mathbf{k+q}}/2)] \times \left(\frac{u_{\mathbf{k+q}}^2 v_{\mathbf{k}}^2}{E_{\mathbf{k}} - E_{\mathbf{k+q}} + i\Omega_\ell} + \frac{v_{\mathbf{k+q}}^2 u_{\mathbf{k}}^2}{E_{\mathbf{k}} - E_{\mathbf{k+q}} - i\Omega_\ell} \right) \Bigg], \quad (3.23)$$

$$B(q) = \frac{1}{2V}\sum_{\mathbf{k}} u_{\mathbf{k+q}} u_{\mathbf{k}} v_{\mathbf{k+q}} v_{\mathbf{k}} \Bigg[[\tanh(\beta E_{\mathbf{k}}/2) + \tanh(\beta E_{\mathbf{k+q}}/2)] \times \left(\frac{1}{E_{\mathbf{k+q}} + E_{\mathbf{k}} + i\Omega_\ell} + \frac{1}{E_{\mathbf{k+q}} + E_{\mathbf{k}} - i\Omega_\ell} \right) + [\tanh(\beta E_{\mathbf{k}}/2) - \tanh(\beta E_{\mathbf{k+q}}/2)] \times \left(\frac{-1}{-E_{\mathbf{k}} + E_{\mathbf{k+q}} - i\Omega_\ell} + \frac{-1}{-E_{\mathbf{k}} + E_{\mathbf{k+q}} + i\Omega_\ell} \right) \Bigg]. \quad (3.24)$$

In terms of these the Gaussian functional integral over the $\hat{b}(x)$ and $\hat{b}^*(x)$ fields finally gives the $1/N$ contribution to the free-energy density

$$f^{(1/N)} = \frac{1}{2\beta V}\sum_q e^{i\Omega_\ell \delta} \ln\{\lambda^2[|A(q)|^2 - |B(q)|^2]\}, \quad (3.25)$$

where the function is evaluated at an imaginary time $\delta = 0^+$.

When inserted into Eqs. (3.1), and using Eq. (3.2), Eq. (3.25) yields an explicit expression for the free-energy density of a resonantly interacting Fermi gas to leading order in $1/N$. We now proceed to compute the effect of the $1/N$ corrections in various limiting regimes, starting with $T=0$.

1. Zero-temperature limit

At zero temperature, the energy density (to subleading order) is

$$\frac{\varepsilon}{N} = \varepsilon^{(0)} + \frac{1}{N}\varepsilon^{(1/N)} + \cdots , \tag{3.26}$$

where $\varepsilon^{(0)} = f^{(0)}(T=0)$ and

$$\varepsilon^{(1/N)} = \frac{1}{2V}\sum_{\mathbf{q}} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} e^{i\Omega\delta} \ln[\lambda^2(|A(\mathbf{q},i\Omega)|^2 - |B(\mathbf{q},i\Omega)|^2)]. \tag{3.27}$$

At large N, the chemical potential μ and gap parameter Δ are very close to their $N\to\infty$ values, within a correction of order $O(1/N)$,

$$\Delta = \Delta_o^{(0)} + \frac{1}{N}\delta\Delta + \cdots , \tag{3.28}$$

$$\mu = \mu_o^{(0)} + \frac{1}{N}\delta\mu + \cdots . \tag{3.29}$$

Inserting Eqs. (3.28) and (3.29) into Eq. (3.27) and using the gap $0 = \partial\varepsilon/\partial\Delta$ and number $nN = -\partial\varepsilon/\partial\mu$ equations, we find that $\delta\Delta$ and $\delta\mu$ satisfy

$$\begin{pmatrix} \partial_{\mu\mu}\varepsilon^{(0)} & \partial_{\mu\Delta}\varepsilon^{(0)} \\ \partial_{\Delta\mu}\varepsilon^{(0)} & \partial_{\Delta\Delta}\varepsilon^{(0)} \end{pmatrix} \begin{pmatrix} \delta\mu \\ \delta\Delta \end{pmatrix} = -\begin{pmatrix} \partial_{\mu}\varepsilon^{(1/N)} \\ \partial_{\Delta}\varepsilon^{(1/N)} \end{pmatrix}, \tag{3.30}$$

where all the derivatives (indicated by the shorthand notation $\partial_{ab} \equiv \frac{\partial}{\partial a}\frac{\partial}{\partial b}$) are evaluated at the saddle point, i.e., $\Delta = \Delta_o^{(0)}$ and $\mu = \mu_o^{(0)}$. Solving for $\delta\Delta$ and $\delta\mu$ yields

$$\begin{pmatrix} \delta\mu \\ \delta\Delta \end{pmatrix} = \frac{-1}{\partial_{\mu\mu}\varepsilon^{(0)}\partial_{\Delta\Delta}\varepsilon^{(0)} - \partial_{\mu\Delta}\varepsilon^{(0)}\partial_{\Delta\mu}\varepsilon^{(0)}} \times \begin{pmatrix} \partial_{\Delta\Delta}\varepsilon^{(0)} & -\partial_{\mu\Delta}\varepsilon^{(0)} \\ -\partial_{\Delta\mu}\varepsilon^{(0)} & \partial_{\mu\mu}\varepsilon^{(0)} \end{pmatrix} \begin{pmatrix} \partial_{\mu}\varepsilon^{(1/N)} \\ \partial_{\Delta}\varepsilon^{(1/N)} \end{pmatrix}, \tag{3.31}$$

thus expressing the shifts $\delta\mu$ and $\delta\Delta$ in the chemical potential and gap in terms of various derivatives of the integrals $\varepsilon^{(0)}$ and $\varepsilon^{(1/N)}$. At unitarity, these can be calculated numerically and are given by

$$\partial_{\mu\mu}\varepsilon^{(0)} = -1.0804n/\epsilon_F, \tag{3.32}$$

$$\partial_{\mu\Delta}\varepsilon^{(0)} = -1.2556n/\epsilon_F, \tag{3.33}$$

$$\partial_{\Delta\Delta}\varepsilon^{(0)} = 1.0804n/\epsilon_F, \tag{3.34}$$

$$\partial_{\mu}\varepsilon^{(1/N)} = -0.542n, \tag{3.35}$$

$$\partial_{\Delta}\varepsilon^{(1/N)} = -0.216n. \tag{3.36}$$

Substituting Eq. (3.31) and the appropriate parameters into Eqs. (3.28) and (3.29) gives the $1/N$ corrections to μ and Δ quoted in Eqs. (1.2) and (1.4) of the Introduction.

2. Near T_c

Next, we consider the vicinity of the transition temperature T_c, for which the saddle-point ($N\to\infty$) results were presented in Sec. III A 2 The $1/N$ corrections to the critical temperature can be evaluated by computing the Thouless criterion ($\partial_\Delta f = 0^+$) and particle number equation from the free energy. Combining the two equations allows the determination of the critical temperature and chemical potential correction. Writing the saddle-point solution with the indices $c^{(0)}$, we can parametrize the $1/N$ corrections to T and μ near T_c via

$$T = T_c^{(0)} + \frac{1}{N}\delta T + \cdots , \tag{3.37}$$

$$\mu = \mu_c^{(0)} + \frac{1}{N}\delta\mu + \cdots , \tag{3.38}$$

analogously to the zero-temperature case in the preceding section. As in that case, the Thouless criterion and particle equation yield a set of two coupled equations,

$$\begin{pmatrix} \partial_{\mu\mu}f^{(0)} & \partial_{\mu T}f^{(0)} \\ \partial_{\Delta\mu}f^{(0)} & \partial_{\Delta T}f^{(0)} \end{pmatrix} \begin{pmatrix} \delta\mu \\ \delta T \end{pmatrix} = -\begin{pmatrix} \partial_{\mu}f^{(1/N)} \\ \partial_{\Delta}f^{(1/N)} \end{pmatrix}, \tag{3.39}$$

where all the derivatives are evaluated at $T = T_c^{(0)}$ and $\mu = \mu_c^{(0)}$ and $\Delta = 0^+$. Solving for δT and $\delta\mu$, we obtain

$$\begin{pmatrix} \delta\mu \\ \delta T \end{pmatrix} = \frac{-1}{\partial_{\mu\mu}f^{(0)}\partial_{\Delta T}f^{(0)} - \partial_{\mu T}f^{(0)}\partial_{\Delta\mu}f^{(0)}} \times \begin{pmatrix} \partial_{\Delta T}f^{(0)} & -\partial_{\mu T}f^{(0)} \\ -\partial_{\Delta\mu}f^{(0)} & \partial_{\mu\mu}f^{(0)} \end{pmatrix} \begin{pmatrix} \partial_{\mu}f^{(1/N)} \\ \partial_{\Delta}f^{(1/N)} \end{pmatrix}. \tag{3.40}$$

At unitarity, we find the matrix elements of Eq. (3.40) to be

$$f^{(0)} = -0.678n\epsilon_F, \tag{3.41}$$

$$\partial_{\mu\mu}f^{(0)} = -1.1720n/\epsilon_F, \tag{3.42}$$

$$\partial_{\mu T}f^{(0)} = -1.2581n/\epsilon_F, \tag{3.43}$$

$$\partial_{\Delta\mu}f^{(0)} = -1.5691n\Delta/\epsilon_F^2, \tag{3.44}$$

$$\partial_{\Delta T}f^{(0)} = 2.3608n\Delta/\epsilon_F^2, \tag{3.45}$$

$$f^{(1/N)} = -0.701n\epsilon_F, \tag{3.46}$$

$$\partial_{\mu}f^{(1/N)} = -2.339n, \tag{3.47}$$

$$\partial_{\Delta}f^{(1/N)} = 2.195n\Delta/\epsilon_F. \tag{3.48}$$

Using Eq. (3.40) together with these matrix elements inside Eqs. (3.37) and (3.38) gives the $1/N$ correction to the critical temperature and the chemical potential quoted in Eqs. (1.8) and (1.9) of the Introduction.

3. Polarized Fermi gas

We now determine the $1/N$ corrections, beyond the mean-field result reviewed in Sec. III A 3, to the upper critical polarization P_{c2}. Recall the two criteria to determine the upper-critical polarization, above which the normal state is

stable. These are given by equality of the superfluid and normal-state free energies (the first-order transition condition) and the normal-state number equation, that are, respectively, given by

$$f_n(\mu,h) = f_s(\mu,h,\Delta), \tag{3.49}$$

$$Nn = -\frac{\partial f_n(\mu,h)}{\partial \mu}. \tag{3.50}$$

Together, these determine the chemical potential difference below which the normal state is unstable to phase separation. In our formulation of the theory, the superfluid-state energy on the right-hand side of Eq. (3.49) is a minimum with respect to a variational parameter, namely Δ,

$$\frac{\partial f_s(\mu,h,\Delta)}{\partial \Delta} = 0. \tag{3.51}$$

Therefore, there are three parameters to solve for, namely Δ, μ, and h, using these three conditions. Once these have been determined, the upper critical polarization follows from $P_{c2} = \frac{n_\uparrow - n_\downarrow}{n_\uparrow + n_\downarrow} = m/n$, where m is the density imbalance. In the superfluid state, the single particle excitations are gapped (for $h < \Delta$). As a consequence, at zero temperature, the susceptibility to the polarization field h vanishes, therefore, the energy of the ground state is independent of h and is given by Eq. (3.26). In the case of the normal state, the result is also relatively simple as the polarization field corresponds to a mere shift of the chemical potential. The $1/N$ correction to the action is

$$S^{(1/N)} = \sum_q \hat{b}^*(q)\Gamma^{-1}(q)\hat{b}(q), \tag{3.52}$$

where

$$\Gamma^{-1}(q) = -\frac{1}{\lambda} + \frac{1}{4V}\sum_{\mathbf{k}} \left(\frac{\sum_{\sigma=\pm} \tanh[\beta(\xi_{\mathbf{k}+\mathbf{q}/2} + \sigma h)/2] + \tanh[\beta(\xi_{\mathbf{k}-\mathbf{q}/2} + \sigma h)/2]}{i\Omega_\ell - \xi_{\mathbf{k}+\mathbf{q}/2} - \xi_{\mathbf{k}-\mathbf{q}/2}} \right). \tag{3.53}$$

The free-energy density contribution due to fluctuations is given by

$$f_n^{(1/N)} = \frac{1}{\beta V}\sum_q e^{i\Omega_\ell \delta} \ln[-\lambda\Gamma^{-1}(q)]. \tag{3.54}$$

The zeroth order $f_n^{(0)}$ result can be obtained by taking the limit $\Delta \to 0$ in Eq. (3.9),

$$f_n^{(0)} = \frac{1}{V}\sum_{\sigma=\pm}\sum_{\mathbf{k}} \left(\xi_{\mathbf{k}} - \frac{2}{\beta} \ln\{2\cosh[\beta(\xi_{\mathbf{k}} + \sigma h)/2]\} \right). \tag{3.55}$$

As in the preceding sections, the parameters μ, h, and Δ acquire corrections, due to Gaussian fluctuations, that are small in $1/N$, and can therefore be written as

$$\Delta = \Delta_{c2}^{(0)} + \frac{1}{N}\delta\Delta + \cdots, \tag{3.56}$$

$$\mu = \mu_{c2}^{(0)} + \frac{1}{N}\delta\mu + \cdots, \tag{3.57}$$

$$h = h_{c2}^{(0)} + \frac{1}{N}\delta h + \cdots. \tag{3.58}$$

Inserting this parametrization into Eqs. (3.49)–(3.51), we find a set of three coupled equations giving these corrections

$$\begin{pmatrix} \partial_\mu f_n^{(0)} - \partial_\mu f_s^{(0)} & \partial_h f_n^{(0)} - \partial_h f_s^{(0)} & \partial_\Delta f_n^{(0)} - \partial_\Delta f_s^{(0)} \\ \partial_{\mu\mu} f_n^{(0)} & \partial_{\mu h} f_n^{(0)} & \partial_{\mu\Delta} f_n^{(0)} \\ \partial_{\Delta\mu} f_s^{(0)} & \partial_{\Delta h} f_s^{(0)} & \partial_{\Delta\Delta} f_s^{(0)} \end{pmatrix} \begin{pmatrix} \delta\mu \\ \delta h \\ \delta\Delta \end{pmatrix} = -\begin{pmatrix} f_n^{(1/N)} - f_s^{(1/N)} \\ \partial_\mu f_n^{(1/N)} \\ \partial_\Delta f_s^{(1/N)} \end{pmatrix}, \tag{3.59}$$

where all the derivatives are evaluated at $\Delta = \Delta_{c2}^{(0)}$, $h = h_{c2}^{(0)}$, and $\mu = \mu_{c2}^{(0)}$. Restricting attention to zero temperature, many of the matrix elements vanish because (1) at unitarity, the superfluid state is gapped and its susceptibility to a "Zeeman" field vanishes, (2) the normal state by definition does not have any superfluid correlations, i.e., its ground-state energy is independent of Δ, (3) the ground-state energy of the superfluid state is a minimum with respect to Δ. We then obtain

$$\begin{pmatrix} \partial_\mu \varepsilon_n^{(0)} - \partial_\mu \varepsilon_s^{(0)} & \partial_h \varepsilon_n^{(0)} & 0 \\ \partial_{\mu\mu} \varepsilon_n^{(0)} & \partial_{\mu h} \varepsilon_n^{(0)} & 0 \\ \partial_{\Delta\mu} \varepsilon_s^{(0)} & 0 & \partial_{\Delta\Delta} \varepsilon_s^{(0)} \end{pmatrix} \begin{pmatrix} \delta\mu \\ \delta h \\ \delta\Delta \end{pmatrix} = -\begin{pmatrix} \varepsilon_n^{(1/N)} - \varepsilon_s^{(1/N)} \\ \partial_\mu \varepsilon_n^{(1/N)} \\ \partial_\Delta \varepsilon_s^{(1/N)} \end{pmatrix}. \tag{3.60}$$

Solving the matrix equation (3.60) yields

$$\delta\mu = \frac{(\varepsilon_s^{(1/N)} - \varepsilon_n^{(1/N)})\partial_{\mu h}\varepsilon_n^{(0)} + \partial_\mu \varepsilon_n^{(1/N)}(\partial_h \varepsilon_n^{(0)})}{(\partial_\mu \varepsilon_n^{(0)} - \partial_\mu \varepsilon_s^{(0)})\partial_{\mu h}\varepsilon_n^{(0)} - (\partial_h \varepsilon_n^{(0)})\partial_{\mu\mu}\varepsilon_n^{(0)}}, \quad (3.61)$$

$$\delta h = \frac{-(\varepsilon_s^{(1/N)} - \varepsilon_n^{(1/N)})\partial_{\mu\mu}\varepsilon_n^{(0)} - \partial_\mu \varepsilon_n^{(1/N)}(\partial_\mu \varepsilon_n^{(0)} - \partial_\mu \varepsilon_s^{(0)})}{(\partial_\mu \varepsilon_n^{(0)} - \partial_\mu \varepsilon_s^{(0)})\partial_{\mu h}\varepsilon_n^{(0)} - (\partial_h \varepsilon_n^{(0)})\partial_{\mu\mu}\varepsilon_n^{(0)}}, \quad (3.62)$$

$$\delta\Delta = -\frac{\partial_\Delta \varepsilon_s^{(1/N)}}{\partial_{\Delta\Delta}\varepsilon_s^{(0)}} + \frac{\partial_{\Delta\mu}\varepsilon_s^{(0)}}{\partial_{\Delta\Delta}\varepsilon_s^{(0)}} \frac{-(\varepsilon_s^{(1/N)} - \varepsilon_n^{(1/N)})\partial_{\mu h}\varepsilon_n^{(0)} - \partial_\mu \varepsilon_n^{(1/N)}(\partial_h \varepsilon_n^{(0)})}{(\partial_\mu \varepsilon_n^{(0)} - \partial_\mu \varepsilon_s^{(0)})\partial_{\mu h}\varepsilon_n^{(0)} - (\partial_h \varepsilon_n^{(0)})\partial_{\mu\mu}\varepsilon_n^{(0)}}. \quad (3.63)$$

The matrix elements, at unitarity, are

$$\partial_\mu \varepsilon_s^{(0)} = -1.7527n, \quad (3.64)$$

$$\partial_{\mu\Delta}\varepsilon_s^{(0)} = -1.5140n/\epsilon_F, \quad (3.65)$$

$$\partial_{\Delta\Delta}\varepsilon_s^{(0)} = 1.3027n/\epsilon_F, \quad (3.66)$$

$$\partial_\mu \varepsilon_n^{(0)} = -n, \quad (3.67)$$

$$\partial_h \varepsilon_n^{(0)} = -0.9326n, \quad (3.68)$$

$$\partial_{\mu\mu}\varepsilon_n^{(0)} = -1.2394n/\epsilon_F, \quad (3.69)$$

$$\partial_{\mu h}\varepsilon_n^{(0)} = -0.6290n/\epsilon_F, \quad (3.70)$$

$$\partial_{hh}\varepsilon_n^{(0)} = -1.2394n/\epsilon_F, \quad (3.71)$$

$$\varepsilon_n^{(1/N)} = -0.0509n\epsilon_F, \quad (3.72)$$

$$\varepsilon_s^{(1/N)} = -0.477n\epsilon_F, \quad (3.73)$$

$$\partial_\mu \varepsilon_n^{(1/N)} = -0.513n, \quad (3.74)$$

$$\partial_h \varepsilon_n^{(1/N)} = 0.452n, \quad (3.75)$$

$$\partial_\Delta \varepsilon_s^{(1/N)} = -0.379n, \quad (3.76)$$

$$\partial_\mu \varepsilon_s^{(1/N)} = -0.948n. \quad (3.77)$$

At this point a few remarks are in order. Some of these matrix elements can be related to each other. Indeed, using scaling arguments akin to those presented in Sec. III A 3, one can show, for instance, $\varepsilon_s^{(1/N)} = \frac{2}{5}\mu\partial_\mu \varepsilon_s^{(1/N)} + \frac{2}{5}\Delta\partial_\Delta \varepsilon_s^{(1/N)}$, as well as $\varepsilon_n^{(1/N)} = \frac{2}{5}\mu\partial_\mu \varepsilon_n^{(1/N)} + \frac{2}{5}h\partial_h \varepsilon_n^{(1/N)}$. Furthermore, we observe that the corrections $\delta\mu$ and δh do not involve any matrix elements containing derivatives with respect to Δ. This merely reflects the fact that Δ is a variational parameter and that one could have reduced the solution for $\delta\mu$ and δh to a set of two coupled equations using Eqs. (3.50). In addition, we point out that the ratio Δ/μ at $h=h_{c2}$ is equal to Δ/μ at $h=0$, a result directly related to the vanishing susceptibility of a gapped superfluid state.

The polarization at the upper-critical field can be written in a $1/N$ expansion as

$$P_{c2} \equiv \frac{m_{c2}}{n_{c2}} = P_{c2}^{(0)} + \frac{1}{N}\delta P, \quad (3.78)$$

where $P_{c2}^{(0)}$ is the result of the $N\to\infty$ calculation and

$$\delta P = P_{c2}^{(0)}\left[\frac{\partial_h \varepsilon_n^{(1/N)}}{\partial_h \varepsilon_n^{(0)}} - \frac{\partial_\mu \varepsilon_n^{(1/N)}}{\partial_\mu \varepsilon_n^{(0)}} + \left(\frac{\partial_{\mu h}\varepsilon_n^{(0)}}{\partial_h \varepsilon_n^{(0)}} - \frac{\partial_{\mu\mu}\varepsilon_n^{(0)}}{\partial_\mu \varepsilon_n^{(0)}}\right)\delta\mu + \left(\frac{\partial_{hh}\varepsilon_n^{(0)}}{\partial_h \varepsilon_n^{(0)}} - \frac{\partial_{\mu h}\varepsilon_n^{(0)}}{\partial_\mu \varepsilon_n^{(0)}}\right)\delta h\right]. \quad (3.79)$$

At unitarity, we find

$$h_{c2}/\epsilon_F = 0.6930 + 0.087/N + O(1/N^2), \quad (3.80)$$

$$P_{c2} = 0.9326 - 0.631/N + O(1/N^2), \quad (3.81)$$

$$\mu_{c2}/\epsilon_F = 0.8586 - 0.458/N + O(1/N^2), \quad (3.82)$$

$$\Delta_{c2}/\epsilon_F = 0.9978 - 0.242/N + O(1/N^2). \quad (3.83)$$

C. Away from unitarity

Although so far we have focused on the problem at the unitarity limit, the same large-N expansion formalism can be used to calculate the properties of a resonant Fermi gas away from the unitarity point. In this section we compute such $1/N$ correction at zero temperature.

The result of Eq. (3.30) holds for any value of the scattering length a_s and therefore can be used away from unitarity. The matrix elements of the left-hand side of Eq. (3.30) can be calculated by taking appropriate derivatives of Eq. (3.2). After straightforward algebraic manipulation, one obtains

$$\partial_{\mu\mu}\varepsilon^{(0)} = -I_5(x_o)\frac{(2m)^{3/2}}{2\pi^2}\Delta_o^{1/2}, \quad (3.84)$$

$$\partial_{\mu\Delta}\varepsilon^{(0)} = -I_6(x_o)\frac{(2m)^{3/2}}{2\pi^2}\Delta_o^{1/2}, \quad (3.85)$$

$$\partial_{\Delta\mu}\varepsilon^{(0)} = \partial_{\mu\Delta}\varepsilon^{(0)}, \quad (3.86)$$

$$\partial_{\Delta\Delta}\varepsilon^{(0)} = I_5(x_o)\frac{(2m)^{3/2}}{2\pi^2}\Delta_o^{1/2}. \quad (3.87)$$

where, following Marini *et al.* [48], we introduced dimensionless function $I_5(x_o)$ and $I_6(x_o)$,

$$I_6(x_o) = \int_0^\infty dx \frac{x^2\xi_x}{E_x^3}, \quad (3.88)$$

$$I_5(x_o) = \int_0^\infty dx \frac{x^2}{E_x^3}, \quad (3.89)$$

and dimensionless parameters

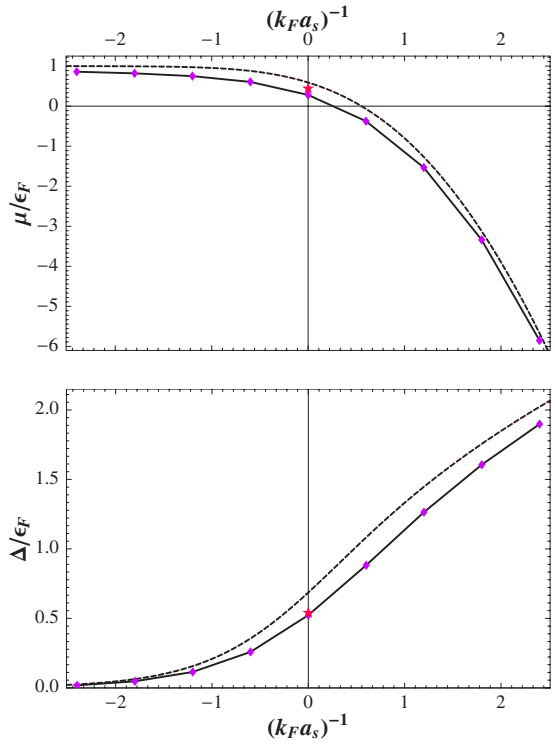

FIG. 1. (Color online) Chemical potential μ and order parameter Δ as a function of $(k_F a_s)^{-1}$. The dashed lines are the $N\to\infty$ results for μ and Δ. The diamond symbols include the $O(1/N)$ corrections, evaluated at $N=1$. The solid lines are a guide to the eye. The star symbols at unitarity are the results of quantum Monte Carlo calculation from Ref. [19].

$$x_o = \frac{\mu_o}{\Delta_o}, \quad x^2 = \frac{1}{\Delta_o}\frac{|\mathbf{k}|^2}{2m},$$

$$\xi_x = \frac{\xi_{\mathbf{k}}}{\Delta_o} = x^2 - x_o, \quad E_x = \frac{E_{\mathbf{k}}}{\Delta_o} = \sqrt{\xi_x^2 + 1}. \tag{3.90}$$

The variables μ_o and Δ_o are the solutions to the gap and particle equations in the $N\to\infty$ limit. For numerical purposes, it is useful to note that Eqs. (3.88) and (3.89) can be expressed in terms of linear combinations of complete elliptic integrals [48].

Substituting previous equations into Eq. (3.31) yields,

$$\begin{pmatrix} \delta\mu \\ \delta\Delta \end{pmatrix} = \frac{2\epsilon_F^{3/2}}{3n\Delta_o^{1/2}} \frac{1}{I_5^2(x_o) + I_6^2(x_o)} \begin{pmatrix} I_5(x_o) & I_6(x_o) \\ I_6(x_o) & -I_5(x_o) \end{pmatrix} \begin{pmatrix} \partial_\mu \varepsilon^{(1/N)} \\ \partial_\Delta \varepsilon^{(1/N)} \end{pmatrix}, \tag{3.91}$$

for the leading order (in $1/N$) corrections $\delta\Delta$ and $\delta\mu$ to the saddle-point (mean-field) result given by Eqs. (3.3) and (3.4) at $T=0$.

Solving Eq. (3.91) numerically yields results displayed in Fig. 1 as a function of $(k_F a_s)^{-1}$.

IV. DISCUSSION AND COMPARISON TO OTHER WORK

In this work we have studied a two-component resonantly interacting Fermi gas in the unitary regime by generalizing the system to a $2N$-component gas with N flavors of spin-$\frac{1}{2}$ fermions, computing a variety of thermodynamic quantities in a systematic expansion in a small parameter $1/N$, with our main results presented in Sec. I B. Although our results are strictly only quantitatively valid to leading order in $1/N$, to compare to experiments and other calculations, in this section we boldly take the $N\to 1$ limit.

TABLE I. Recent experimental results for ξ compared with calculated values (compilation taken from Ref. [29] with a few additional results).

		ξ
Experimental results	Ghem *et al.* [50]	0.74(7)
	Bartenstein *et al.* [4]	$0.32^{+0.13}_{-0.10}$
	Bourdel *et al.* [5]	0.36(15)
	Duke [51]	0.51(4)
	Partridge *et al.* [41]	0.46(5)
	Regal *et al.* [54]	0.38(7)
	Stewart *et al.* [52]	$0.46^{+0.05}_{-0.12}$
Calculated values	Mean field	$\xi<0.5906$
	Astrakharchik *et al.* [20]	0.42(1)
	Carlson *et al.* [19]	0.44(1)
	Perali *et al.* [27]	0.455
	Padé approximation [18,26]	0.33
	Haussmann *et al.* [29]	0.36
	Epsilon expansion [24]	0.30 to 0.37
	This work	0.28

The study of the unitarity point can be seen as a benchmark for many-body theories. In this section, we collect the various estimates to thermodynamic quantities at the unitarity point from the literature and compare to our results. Starting with BCS expressions, a mean-field estimate of $\xi=0.59$ can be obtained. Since this approach is based on a variational wave function, it provides a strict upper bound. Pade approximant techniques have also been applied to a Fermi gas expansion in terms of $k_F a_s$ to extract an estimate for $\xi=0.326$ [18,26]. Perali *et al.* [27] have introduced a diagrammatic method based on the t-matrix extended to the superfluid phase finding, at unitarity, $\xi=0.455$. Most recently, Haussmann *et al.* [29] proposed a self-consistent and conserving theory to study the crossover and found $\xi=0.36$. An epsilon expansion of the unitarity gas, based on an expansion from the dimension $d=4-\epsilon$ has been performed to second loop order [24]. Depending on the Borel-Pade extrapolation schemes used, the results range from $\xi=0.30$ to $\xi=0.37$.

At present time, the best estimates for ξ coming from fixed node Green's function Monte Carlo calculations [19,20] yield $\xi\approx 0.44\pm 0.01$ (by Carlson *et al.* [19]) and $\xi \approx 0.42\pm 0.01$ by Astrakharchik *et al.* [20]. The nodal method is based on a variational approach, thus also giving a strict upper bound for ξ (Table I).

There has also been a large experimental effort aimed at extracting the parameter ξ. Experimental measurements of

TABLE II. Universal ratio $\Delta_{\rm exc}/\epsilon_F$ at unitarity: Comparison between numerical and theoretical approaches.

	$\Delta_{\rm exc}/\epsilon_F$
Mean field	0.6864
Carlson *et al.* (Monte Carlo) [19]	0.54
Haussmann *et al.* [29]	0.46
Nishida *et al.* [23] (epsilon expansion)	0.60
This work	0.49

the expansion of ^{6}Li in the unitarity limit from a harmonic trap have determined ξ (often quoted as $\beta \equiv \xi - 1$). Experimental results for ξ were first obtained by Ghem *et al.* [50], $\xi=0.74(7)$, Bartenstein *et al.* [4] $\xi=0.32^{+10}_{-13}$, and Bourdel *et al.* [5], $\xi=0.36(15)$. Most recently, the Duke group [51] obtained $\xi=0.51(4)$, whereas the Rice group [41] found $\xi=0.46(5)$. Experiments on ^{40}K by the Boulder group [52] yielded $\xi=0.46^{+0.05}_{-0.12}$. Our prediction for $\xi=0.28$ at $N=1$ is qualitatively consistent with these results, reflecting the smallness of fluctuations around the saddle-point mean-field solution at $T=0$.

Another important quantity is the single particle excitation gap at unitarity. In the Green's function Monte Carlo calculations of Refs. [19,53], an estimate for the spectroscopic energy gap $\Delta_{\rm exc}$ was determined from the odd-even staggering of the total energy as the number of particles is increased and found $\Delta_{\rm exc}=0.54\epsilon_F$. It is not *a priori* clear whether this gap is generally identical to the expectation value $\Delta \equiv |\lambda \langle \psi_\uparrow \psi_\downarrow \rangle|$; in the Appendix we show that they are equal to zeroth order but differ at the $O(1/N)$ computed here. Our prediction for $\Delta_{\rm exc}/\epsilon_F$, evaluated at $N=1$, is in good agreement with the Monte Carlo calculations as seen in Table II.

The critical temperature at unitarity has also been investigated. The $1/N$ calculations shows relatively large correction to the $N\to\infty$ result, so that the $N\to 1$ limit of Eq. (1.8) is clearly not sensible *quantitatively* but can be qualitatively interpreted as predicting a large negative correction to the mean-field result for T_c. This strong correction to the saddle-point approximation is expected and reflects the fact that the mean-field solution neglects the effect of bound pairs and describes the normal state as consisting of free fermions [31]. Quantum Monte calculations performed by Bulgac *et al.* [21] have indeed determined a relatively low critical temperature [$T_c=0.23(2)\epsilon_F$], and Burovski *et al.* [22] have arrived to $T_c=0.152(7)\epsilon_F$ using diagrammatic determinant Monte Carlo. We note that this result is below the BEC limit, $T_{\rm BEC}=0.218\epsilon_F$, in contrast to the earlier work of Nozieres and Schmitt-Rink [31]. Burovski *et al.* also give $\mu=0.493(14)\epsilon_F$ and $\epsilon=0.31(1)\epsilon_F$ at T_c. Finally, we point out a recent $1/N$ calculation by Nikolić and Sachdev [55], that gives a result for $1/T_c$, that is consistent with our prediction for T_c.

Now we turn to the polarized Fermi gas. We have determined the $1/N$ correction to the upper critical field P_{c2}. Besides the mean-field value, we do not know of any other theoretical estimates for P_{c2}. This is partly due to the difficulty of Monte Carlo calculations to tackle the sign problem for a polarized Fermi gas. However, a recent experiment on ^{6}Li has found P_{c2} to be below its mean-field value estimate, and given by $P_{c2}=0.70$ [40]. This is consistent with the calculation of the $1/N$ correction that shows that P_{c2} decreases with decreasing N going from its mean-field value $P_{c2}^{N=\infty}=0.933$. The naive substitution $N=1$ in Eq. (3.81) yields $P_{c2}^{N=1}=0.302$, which underestimates the experimental result.

V. SUMMARY

To summarize, we have studied a resonant Fermi gas interacting via short-range attractive interactions by generalizing the model to N fermion flavors and employing a theoretical method that is perturbative in $1/N$. The $1/N$ expansion provides a systematic scheme for quantitatively determining corrections to the standard BEC-BCS mean-field theory of interacting Fermi gases.

Although our primary goal was the computation of various quantities to leading order in $1/N$ in the vicinity of the unitarity point (where universality holds), we also computed the zero-temperature gap and chemical potential away from the unitary point. Clearly, future work must be done to generalize our other results for arbitrary coupling (i.e., Feshbach resonance detuning) as well as to compute the next $O(1/N^2)$ term.

Note added. We learned of an independent, and closely related $1/N$-expansion study by Nikolić and Sachdev [55]. Where there is overlap, our results are in agreement with theirs.

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APPENDIX: PROPERTIES OF GREEN'S FUNCTIONS

In this Appendix we determine properties of the single-particle excitation spectrum. In particular, we calculate the single-particle excitation gap $\Delta_{\rm exc}$ to subleading order in $1/N$. Although $\Delta_{\rm exc}$ (which is more commonly measured in experiments) is equal to the order parameter Δ within mean-field theory (i.e., at $N\to\infty$), beyond mean-field theory they are slightly different.

The gap in the single-particle excitation spectrum is determined as the lowest energy state of the single-particle Green's function, defined as

$$\mathcal{G}(x)_{\sigma,\sigma'} = \langle \psi_{1,\sigma}(x) \bar{\psi}_{1,\sigma'}(0) \rangle, \tag{A1}$$

where without loss of generality, we selected the flavor $i=1$. To calculate this expectation value within our formalism, it is useful to introduce source fields in the action

$$S_{\text{source}} = \sum_{i=1}^{N} \int_0^\beta d\tau \int d^3\mathbf{r}[\Phi_i(x)\bar{\Psi}_i(x) + \bar{\Phi}_i(x)\Psi_i(x)], \tag{A2}$$

where we introduced a Nambu representation for fermions, i.e., $\bar{\Psi}_i(x)=(\bar{\psi}_{i\uparrow}(x), \psi_{i\downarrow}(x))$ and where $\bar{\Phi}_i(x)=(\bar{\eta}_{i\uparrow}(x), \eta_{i\downarrow}(x))$ are the source fields. From the auxiliary fields, we derive

$$\mathcal{G}(x)_{\sigma,\sigma'} = \left. \frac{\partial^2 \ln Z[\bar{\eta},\eta]}{\partial\eta_{1,\sigma'}(0)\partial\bar{\eta}_{1,\sigma}(x)} \right|_{\bar{\eta}=\eta=0}. \tag{A3}$$

Using the formula

$$\int D\bar{\Psi}_i(x)D\Psi_i(x)e^{-\Sigma_{ix}[\bar{\Psi}_i(x)M(x)\Psi_i(x)+\Phi_i(x)\bar{\Psi}_i(x)+\bar{\Phi}_i(x)\Psi_i(x)]}$$

$$= \text{Det}(M)e^{\Sigma_{ix}\bar{\Phi}_i(x)M^{-1}(x)\Phi_i(x)}, \tag{A4}$$

we find the partition function is given by

$$Z[\bar{\Phi},\Phi] = Z_b^{-1}\int Db(x)Db^*(x)e^{-S[b]-S_s[\Phi]}, \tag{A5}$$

where

$$S_s[\Phi] = -\sum_{i=1}^{N}\int_0^\beta d\tau \int d^3\mathbf{r}\bar{\Phi}_i(x)G(x)\Phi_i(x), \tag{A6}$$

and $S(b)$ is given by Eq. (2.10). Using, Eq. (A3), we arrive to

$$\mathcal{G}(x) = \langle [G(x)]_{11}\rangle_{S(b)}, \tag{A7}$$

where the Green's function matrix is given by

$$G^{-1}(x) = \begin{pmatrix} -\partial_\tau + \dfrac{\nabla^2}{2m} + \mu & b(x) \\ b^*(x) & -\partial_\tau - \dfrac{\nabla^2}{2m} - \mu \end{pmatrix}. \tag{A8}$$

Using the decomposition given in Eq. (2.13), we find to subleading order

$$[\langle G(k)\rangle_{S(b)}]^{-1} = G_{(0)}^{-1}(k) - \Sigma(k), \tag{A9}$$

where

$$\Sigma(k) = \frac{1}{N}\sum_{k'}\langle G_{(1)}^{-1}(k-k')G_{(0)}(k')G_{(1)}^{-1}(k'-k)\rangle_{S^{(1/N)}}. \tag{A10}$$

In particular, we find for the self-energy

$$\Sigma(k) = \frac{1}{N}\sum_{k'}\frac{1}{|A(k-k')|^2 - |B(k-k')|^2}\frac{1}{\omega'^2 + E_{\mathbf{k}'}^2} \times \begin{pmatrix} -A(k'-k)(i\omega' - \xi_{\mathbf{k}'}) & -B(k-k')\Delta^* \\ -B^*(k-k')\Delta & -A(k-k')(i\omega' + \xi_{\mathbf{k}'}) \end{pmatrix}. \tag{A11}$$

By solving the equation $\det[\langle G(\mathbf{k},\omega)\rangle^{-1}]_{i\omega\to\omega}=0$, we obtain the dispersion relation. At leading order, i.e., $N\to\infty$, this is given by

$$\omega_{\mathbf{k}}^{(0)} = \sqrt{\xi_{\mathbf{k}}^2 + (\Delta_o^{(0)})^2}. \tag{A12}$$

Since the chemical potential is positive, it is possible to find a wave vector such that the condition $\xi_{\mathbf{k}}=0$ is fulfilled, giving, at leading order, that the lowest energy excitation is equal to the order parameter, i.e., $\Delta_{\text{exc}}=\Delta_o^{(0)}$. At subleading order, the additional contribution due to the self-energy breaks this equality. In this case, we determine the excitation gap by expanding the dispersion relation around its minimum, namely $\epsilon_{\mathbf{k}}=\mu$. Straightforward manipulations give

$$\Delta_{\text{exc}} = \Delta + \tfrac{1}{2}(\Sigma_{11}+\Sigma_{22}-2\Sigma_{12})\Big|_{\substack{|\mathbf{k}|=\sqrt{2m\mu_o^{(0)}} \\ i\omega=\Delta_o^{(0)}}}, \tag{A13}$$

where Δ is the order parameter calculated at order $1/N$. Numerical evaluation of the self-energy gives $\Sigma_{11}+\Sigma_{22}-2\Sigma_{12}=-0.067\epsilon_F$, yielding

$$\Delta_{\text{exc}}/\epsilon_F = 0.6864 - 0.196/N + O(1/N^2), \tag{A14}$$

the final result.

[1] C. A. Regal, M. Greiner, and D. S. Jin, Phys. Rev. Lett. **92**, 040403 (2004).

[2] M. W. Zwierlein, C. A. Stan, C. H. Schunck, S. M. F. Raupach, A. J. Kerman, and W. Ketterle, Phys. Rev. Lett. **92**, 120403 (2004).

[3] J. Kinast, S. L. Hemmer, M. E. Gehm, A. Turlapov, and J. E. Thomas, Phys. Rev. Lett. **92**, 150402 (2004).

[4] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. H. Denschlag, and R. Grimm, Phys. Rev. Lett. **92**, 203201 (2004).

[5] T. Bourdel, L. Khaykovich, J. Cubizolles, J. Zhang, F. Chevy, M. Teichmann, L. Tarruell, S. J. J. M. F. Kokkelmans, and C. Salomon, Phys. Rev. Lett. **93**, 050401 (2004).

[6] G. B. Partridge, K. E. Strecker, R. I. Kamar, M. W. Jack, and R. G. Hulet, Phys. Rev. Lett. **95**, 020404 (2005).

[7] C. Chin, M. Bartenstein A. Altmeyer, S. Riedl, S. Jochim, J. H. Denschlag, and R. Grimm, Science **305**, 1128 (2004).

[8] M. W. Zwierlein, J. R. Abo-Shaeer, A. Schirotzek, C. H. Schunck, and W. Ketterle, Nature (London) **435**, 1047 (2005).

[9] D. M. Eagles, Phys. Rev. **186**, 456 (1969).

[10] A. J. Leggett, in *Modern Trends in the Theory of Condensed Matter*, edited by A. Pekalski and J. Przystawa, Lecture Notes in Physics, Vol. 115 (Springer-Verlag, Berlin, 1980), p. 13.

[11] D. S. Petrov, C. Salomon, and G. V. Shlyapnikov, Phys. Rev. Lett. **93**, 090404 (2004); Phys. Rev. A **71**, 012708 (2005).

[12] I. V. Brodsky, M. Yu Kagan, A. V. Klaptsov, R. Combescot, and X. Leyronas, Phys. Rev. A **73**, 032724 (2006).

[13] J. Levinsen and V. Gurarie, Phys. Rev. A **73**, 053607 (2006).

[14] V. Gurarie and L. Radzihovsky, Ann. Phys. (N.Y.) **322**, 2 (2007).

[15] A. V. Andreev, V. Gurarie, and L. Radzihovsky, Phys. Rev. Lett. **93**, 130402 (2004).
[16] In the case of s-wave pairing, the typical experiments utilize resonances that are very broad, such that the effective range parameter r_0 satisfies $|r_0|n^{1/3} \ll 1$. It follows that the effective range essentially drops out of the problem, and the system is characterized by a single gas parameter $a_s n^{1/3}$. As discussed in Refs. [14,15], in the opposite narrow-resonance regime, there is a natural small parameter $\propto (k_F|r_0|)^{-1}$ allowing a systematic perturbative study of resonantly interacting Fermi gases even at the unitarity point and throughout the BEC-BCS crossover.
[17] T.-L. Ho, Phys. Rev. Lett. **92**, 090402 (2004).
[18] H. Heiselberg, Phys. Rev. A **63**, 043606 (2001).
[19] J. Carlson, S.-Y Chang, V. R. Pandharipande, and K. E. Schmidt, Phys. Rev. Lett. **91**, 050401 (2003).
[20] G. E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, Phys. Rev. Lett. **93**, 200404 (2004).
[21] A. Bulgac, J. E. Drut, and P. Magierski, Phys. Rev. Lett. **96**, 090404 (2006).
[22] E. Burovski, N. Prokof'ev, B. Svistunov, and M. Troyer, Phys. Rev. Lett. **96**, 160402 (2006).
[23] Y. Nishida and D. T. Son, Phys. Rev. Lett. **97**, 050403 (2006); Y. Nishida, e-print cond-mat/0608321.
[24] P. Arnold, J. E. Drut, and D. T. Son, e-print cond-mat/0608477.
[25] Y. Nishida and D. T. Son, e-print cond-mat/0607835.
[26] G. A. Baker, Phys. Rev. C **60**, 054311 (1999).
[27] A. Perali, P. Pieri, and G. C. Strinati, Phys. Rev. Lett. **93**, 100404 (2004).
[28] H. Hu, X.-J. Liu, and P. D. Drummond, Phys. Rev. A **73**, 023617 (2006); Europhys. Lett. **74**, 574 (2006).
[29] R. Haussmann, W. Rantner, S. Cerrito, and W. Zwerger, Phys. Rev. A **75**, 023610 (2007).
[30] S. Sachdev and Z. Wang, Phys. Rev. B **43**, 10229 (1991).
[31] P. Nozières and S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195 (1985).
[32] M. Drechsler and W. Zwerger, Ann. Phys. **1**, 15 (1992).
[33] C. A. R. Sá de Melo, M. Randeria, and J. R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).
[34] R. Haussmann, Z. Phys. B: Condens. Matter **91**, 291 (1993); Phys. Rev. B **49**, 12975 (1994).
[35] M. E. Fisher, Rev. Mod. Phys. **46**, 597 (1974); K. G. Wilson, *ibid.* **47**, 773 (1975).
[36] Z. Nussinov and S. Nussinov, Phys. Rev. A **74**, 053622 (2006).
[37] E. Brezin and D. J. Wallace, Phys. Rev. B **7**, 1967 (1973).
[38] S. K. Ma, Phys. Rev. A **10**, 1818 (1974).
[39] M. Moshe and J. Zinn-Justin, Phys. Rep. **385**, 69 (2003).
[40] M. W. Zwierlein, A. Schirotzek, C. H. Schunck, and W. Ketterle, Science **311**, 492 (2006).
[41] G. B. Partridge, W. Li, R. I. Kamar, Y.-A. Liao, and R. G. Hulet, Science **311**, 503 (2006).
[42] Y. Shin, M. W. Zwierlein, C. H. Schunck, A. Schirotzek, and W. Ketterle, Phys. Rev. Lett. **97**, 030401 (2006).
[43] D. E. Sheehy and L. Radzihovsky, Phys. Rev. Lett. **96**, 060401 (2006); e-print cond-mat/0607803 Ann. Phys. (to be published).
[44] P. F. Bedaque, H. Caldas, and G. Rupak, Phys. Rev. Lett. **91**, 247002 (2003).
[45] H. Caldas, Phys. Rev. A **69**, 063602 (2004).
[46] M. M. Parish, F. M. Marchetti, A. Lamacraft, and B. D. Simons, Nat. Phys. **3**, 124 (2007).
[47] To make the $Sp(2N)$ invariance of the N flavor model explicit, we write the fermions in a column vector of length $2N$ as $\Psi^T = (\psi_{1\downarrow} \cdots \psi_{N\downarrow} \psi_{1\uparrow} \cdots \psi_{N\uparrow})$, which transforms Eq. (2.4) into a form $H = \int d^3\mathbf{r} \Psi(\mathbf{r})^\dagger \left(-\frac{\nabla_\mathbf{r}^2}{2m} - \mu\right) \Psi(\mathbf{r}) + \frac{\lambda}{4N} \int d^3\mathbf{r} (\Psi^T \Omega \Psi)^\dagger (\Psi^T \Omega \Psi)$ with $\Omega = \begin{pmatrix} 0 & \hat{I} \\ -\hat{I} & 0 \end{pmatrix}$ an antisymmetric matrix and $\hat{I}$ the $N \times N$ identity matrix.
[48] M. Marini, F. Pistolesi, and G. C. Strinati, Eur. Phys. J. B **1**, 151 (1998).
[49] E. Burovski, N. Prokof'ev, B. Svistunov, and M. Troyer, New J. Phys. **8**, 153 (2006).
[50] M. E. Gehm, S. L. Hemmer, S. R. Granade, K. M. O'Hara, and J. E. Thomas, Phys. Rev. A **68**, 011401(R) (2003).
[51] J. Kinast, A. Turlapov, J. E. Thomas, Q. Chen, J. Stajic, and K. Levin, Science **307**, 1296 (2005).
[52] J. T. Stewart, J. P. Gaebler, C. A. Regal, and D. S. Jin, Phys. Rev. Lett. **97**, 220406 (2006).
[53] S. Y. Chang, V. R. Pandharipande, J. Carlson, and K. E. Schmidt, Phys. Rev. A **70**, 043602 (2004).
[54] C. A. Regal, M. Greiner, S. Giorgini, M. Holland, and D. S. Jin, Phys. Rev. Lett. **95**, 250404 (2005).
[55] P. Nikolić and S. Sachdev, Phys. Rev. A **75**, 033608 (2007).